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UNSTEADY NONLINEAR PROBLEM OF THE HORIZONTAL MOTION OF A CONTOUR UNDER THE INTERFACE BETWEEN TWO LIQUIDS

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A method of solving the initial boundary-value problem of the horizontal motion of a circular cylinder under the interface between two liquids is developed within the framework of nonlinear theory and implemented numerically. Profiles of generated waves and hydrodynamic loads are calculated for the problem of the acceleration of a circular cylinder under the free surface of a heavy liquid. The phenomenon of wave breaking is considered in detail.

Unsteady nonlinear problems of the generation of surface and internal waves by a body moving in a liquid are the subject of extensive research. This interest in these problems is motivated by the possibility of modeling complex wave flows and solving a number of practical problems. In particular, the solution of the problem of the acceleration of a contour at the interface of media in a complete nonlinear formulation allows one to investigate breaking of waves behind the body. Advances in this area are related to the development of numerical methods. The latter are reviewed most comprehensively by Yeung [1] and Romate [2], who classify the available methods of solving wave problems, discuss features of their use, estimate the effectiveness of the corresponding algorithms, and report some results of wave-flow calculations. Sturova [3] presents calculation results for plane gravity waves generated by various disturbances, including those produced by a moving body. Chen and Vorus [4] solved the problems of the motion of a circular cylinder above and under the water–air interface (the solutions were constructed in Lagrangian coordinates using potential theory) and reported results of calculations of wave profiles produced by motion of a cylinder from the quiescent state. Kim and Hwang [5] studied the nonlinear unsteady problem of the horizontal motion of a lifting profile under a free surface. The solution was constructed by the boundary element method in the spectral formulation. Extensive results of calculation for the shape of the free surface and hydrodynamic characteristics of the profile are given.

In the present paper, we propose a numerical method of solving the problem of the horizontal motion of a contour at the interface between media. A numerical experiment on the evaluation of the effect of the problem's parameters on the flow characteristics is performed. The process of wave breaking is investigated in detail. The hydrodynamic responses acting on the contour are calculated.

1. We consider the problem of the horizontal motion from the quiescent state of contour $L_0(t)$ under liquid interface $L_1(t)$. In the lower D_1 and upper D_2 layers, the liquid is ideal, incompressible, and homogeneous. The coordinate system is introduced so that the x axis coincides with the undisturbed interface $L_1(0)$. At the initial time, the center of the cylinder is at the point with coordinates $(0, -h)$. In the chosen coordinate system, the velocity vector of the cylinder $\boldsymbol{V}_{L_0}(t) = (V_{L_0x}, V_{L_0y})$ has the form

$$\boldsymbol{V}_{L_0}(t) = \begin{cases} (-U_0 t/T, 0), & 0 \leqslant t \leqslant T, \\ (-U_0, 0), & t > T, \end{cases}$$

which corresponds to acceleration by a linear law from the quiescent state to a certain constant velocity.

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The velocity potential $\varphi_k(x,y,t)$ that describes motion of the liquid in the region D_k satisfies the Laplace equation

$$\Delta\varphi_k(x,y,t)=0,\qquad (x,y)\in D_k(t)\backslash L_0(t),\qquad k=1,\,2. \tag{1.1}$$

At the interface, the following kinematic and dynamic conditions, written for points $L_1(t)$ moving at velocity $\nabla\varphi_0(x,y,t)$, are satisfied:

$$\nabla\varphi_1(x,y,t)\cdot\boldsymbol{n}_1=\nabla\varphi_2(x,y,t)\cdot\boldsymbol{n}_1=\frac{\partial\boldsymbol{r}_1}{\partial t}\cdot\boldsymbol{n}_1,\qquad (x,y)\in L_1(t); \tag{1.2}$$

$$\begin{gathered}\rho_1\frac{\partial\varphi_1(x,y,t)}{\partial t}-\rho_2\frac{\partial\varphi_2(x,y,t)}{\partial t}+\rho_1\frac{(\nabla\varphi_1(x,y,t))^2}{2}-\rho_2\frac{(\nabla\varphi_2(x,y,t))^2}{2}\\ -\rho_1\nabla\varphi_0(x,y,t)\nabla\varphi_1(x,y,t)+\rho_2\nabla\varphi_0(x,y,t)\nabla\varphi_2(x,y,t)+(\rho_1-\rho_2)gy(x,t)=0,\\ (x,y)\in L_1(t);\end{gathered} \tag{1.3}$$

the surface of the contour $L_0(t)$ obeys the nonpenetration condition

$$(\nabla\varphi_1(x,y,t)-\boldsymbol{V}_{L_0}(t))\cdot\boldsymbol{n}_0=0,\qquad (x,y)\in L_0(t). \tag{1.4}$$

In (1.2)–(1.4), $\boldsymbol{r}_1$ is the radius-vector of the point $(x,y)\in L_1(t)$, $\boldsymbol{n}_j$ is the normal to $L_j(t)$ at the point $(x,y)\in L_j(t)$ $(j=0,\,1)$, ρ_k is the liquid density in the kth layer, g is the acceleration of gravity, and $\partial\,/\partial t$ is the derivative calculated in a moving coordinate system.

At infinitely far points D_1 and D_2, the conditions of absence of disturbances of the velocities and the interface are satisfied:

$$\lim_{(x,y)\to\pm\infty}\nabla\varphi_k(x,y,t)=0,\qquad k=1,\,2; \tag{1.5}$$

$$\lim_{x\to\pm\infty}y=0,\qquad (x,y)\in L_1(t). \tag{1.6}$$

The initial conditions for the interface and the potential have the form

$$L_1(0):\ \ y=0,\qquad x\in(-\infty,+\infty),\qquad \nabla\varphi_k(x,y,0)=0,\qquad (x,y)\in D_k(0)\backslash L_0(0). \tag{1.7}$$

2. We reduce the initial boundary-value problem (1.1)–(1.7) for the velocity potentials $\varphi_k(x,y,t)$ to a system of integrodifferential equations for the intensities of the singularities modeling the liquid and solid boundaries. For this, we place a vortex sheet with intensity $\gamma_1(s_1,t)$ $(\gamma_1(\pm\infty,t)=0)$ along the contour $L_1(t)$ and a layer of sources $q(s_0,t)$ along $L_0(t)$. Then, the complex velocity of disturbed motion of the liquid in the regions D_k $(k=1,\,2)$ has the form

$$\overline{V}(z,t)=\frac{1}{2\pi i}\int\limits_{L_1(t)}\frac{\gamma_1(s_1,t)\,ds_1}{z-\zeta(s_1)}+\frac{1}{2\pi}\int\limits_{L_0(t)}\frac{q(s_0,t)\,ds_0}{z-\zeta(s_0)}; \tag{2.1}$$

$$\lim_{x\to\pm\infty}\overline{V}(z,t)=0. \tag{2.2}$$

We adopt the following assumption on the velocity $\nabla\varphi_0(x,y,t)$ $[(x,y)\in L_1(t)]$:

$$\frac{\partial\varphi_0(x,y,t)}{\partial x}-i\frac{\partial\varphi_0(x,y,t)}{\partial y}=\overline{V}_1(z(s_1),t),\qquad \overline{V}_j(z(s_1),t)=\overline{V}(z(s_j),t),\quad j=0,\,1. \tag{2.3}$$

We note that for $z(s_j)\in L_j(t)$ $(j=0,\,1)$, the improper integrals entering expression (2.1) should be understood in the sense of the Cauchy principal value.

With allowance for (2.3), we write boundary conditions (1.2)–(1.4) as

$$\frac{\partial z(s_1)}{\partial t}=V_1(z(s_1),t),\qquad z(s_1)\in L_1(t); \tag{2.4}$$

$$\frac{\partial G(s_1,t)}{\partial t}=\rho_*\Big(\frac{V_1(z(s_1),t)\overline{V}_1(z(s_1),t)}{2}-g\,\mathrm{Im}\,z(s_1)-\frac{\gamma_1^2(s_1,t)}{8}\Big),\qquad z(s_1)\in L_1(t),$$

$$G(s_1,t)=\int\limits_{-\infty}^{s_1}\left(\frac{\gamma_1(\sigma_1,t)}{2}+\rho_*V_{1s}(\sigma_1,t)\right)d\sigma_1, \tag{2.5}$$

$$V_{js}(s_j,t)=\operatorname{Re}\left(V_j(z(s_j),t)\exp\left(i\theta_j(s_j,t)\right)\right),\qquad z(s_j)\in L_j(t),\quad j=0,\,1;$$

$$\begin{gathered}\frac{q(s_0,t)}{2}=\operatorname{Im}\left((\overline{V}_0(z(s_0),t)-\overline{V}_{L_0}(t))\exp(i\theta_0(s_0,t))\right),\\ z(s_0)\in L_0(t),\qquad V_{L_0}(t)=V_{L_0x}+iV_{L_0y},\end{gathered} \tag{2.6}$$

where $\theta_j(s_j,t)$ is the angle between the tangent to the point $z(s_j)\in L_j(t)$ ($j=0,\,1$) and the x axis; $\rho_*=(\rho_1-\rho_2)/(\rho_1+\rho_2)$.

Thus, the initial boundary-value problem (1.1)–(1.7) is reduced to determining the functions $\gamma_1(s_1,t)$ and $q(s_0,t)$ and the interface $L_1(t)$ from the integrodifferential relations (2.4)–(2.6) with allowance for (2.1), (2.2), and initial conditions of the form

$$\operatorname{Im}z=0,\qquad z\in L_1(0),\qquad \gamma_1(s_1,0)=q(s_0,0)=0.$$

The pressure distribution over the contour $p(s_0,t)$, and the overall hydrodynamic loads R_x and R_y and the moment M about the point $z_M=x_M+iy_M$ are defined by the formulas

$$\begin{gathered}p(s_0,t)-f(t)=-\rho_1\left[\frac{\partial}{\partial t}\int\limits_0^{s_0}V_{0s}(\sigma_0,t)\,d\sigma_0-\operatorname{Re}\left(\overline{V}_L(t)V_0(z(s_0),t)\right)+\frac{\overline{V}_0(z(s_0),t)V_0(z(s_0),t)}{2}\right],\\ R_x-iR_y=i\int\limits_{L_0(t)}(p(s_0,t)-f(t))\exp\left(-i\theta_0(s_0,t)\right)ds_0,\\ M=-\int\limits_{L_0(t)}(p(s_0,t)-f(t))[(\xi(s_0,t)-x_M)\cos\theta_0(s_0,t)+(\eta(s_0,t)-y_M)\sin\theta_0(s_0,t)]\,ds_0,\end{gathered} \tag{2.7}$$

where $f(t)$ is a function that depends only on time.

3. The resulting system of integrodifferential equations (2.4)–(2.6) is nonlinear. This is due to two factors: the unknown functions $\gamma_1(s_1,t)$ and $q(s_0,t)$ enter boundary condition (2.5) in a nonlinear manner and the shape of the interface $L_1(t)$ is unknown. In this connection, there are certain difficulties in solving the system obtained.

We solve system (2.4)–(2.6) by the collocation method. In each time step t_n ($n=1,\,2,\ldots$), we consider the interface L_1^n in a finite interval (the superscript denotes the function determined in the nth time step). We divide the contours L_1^n into intervals $[s_{1i-1}^n,s_{1i}^n]$ ($i=1,\ldots,I$) and L_0^n into $[s_{0j-1},s_{0j}]$ ($j=1,\ldots,J$). In these intervals, we choose collocation points $z^n(s_{1i}^{n*})\in L_1^n$ ($s_{1i}^{n*}\in[s_{1i-1}^n,s_{1i}^n]$) and $z^n(s_{0j}^*)\in L_0^n$ ($s_{0j}^*\in[s_{0j-1},s_{0j}]$). We require that (2.4) and (2.5) be satisfied at the points $z^n(s_{1i}^{n*})$ ($i=1,\ldots,I$) and the nonpenetration condition on the contour (2.6) be satisfied at the points $z^n(s_{0j}^*)$ ($j=1,\ldots,J$). The system is then solved by two iterative procedures. One of these involves integration with respect to time of Eqs. (2.4) and (2.5) using an explicit scheme. In this case, in each time step t_n ($n=1,\,2,\ldots$), we obtain the value of the function $G^n(s_{1i}^{n*})$ and the shape of the interface $z^n(s_{1i}^{n*})\in L_1^n$. The other iterative procedure is employed to solve, in each time step, the system of linear algebraic equations obtained by discretization of the relations

$$\frac{\gamma_1^n(s_{1i}^{n*})}{2}+\rho_*V_{1s}^n(s_{1i}^{n*})=\frac{\partial G^n(s_{1i}^{n*})}{\partial s_1^n},\qquad z^n(s_{1i}^{n*})\in L_1^n,\quad i=1,\ldots,I,$$

$$\frac{q^n(s_{0j}^*)}{2}=\operatorname{Im}\left((\overline{V}_0^n(z^n(s_{0j}^*))-\overline{V}_{L_0}^n)\exp(i\theta_0^n(s_{0j}^*))\right),\qquad z^n(s_{0j}^*)\in L_0^n,\quad j=1,\ldots,J.$$

The discretization is performed by the high-order slab method [6]. For this, the interface L_1^n in the ith interval $[s_{1i-1}^n,s_{1i}^n]$ ($i=1,\ldots,I$) and the contour L_0^n in the jth interval $[s_{0j-1},s_{0j}]$ ($j=1,\ldots,J$) are approximated by a parabola, and $\gamma_1^n(s_1^n)$ and $q^n(s_0)$ in the same intervals are approximated by a linear function. Solving the system of linear algebraic equations for the values of the functions $\gamma_1^n(s_1^n)$ and $q^n(s_0)$ at

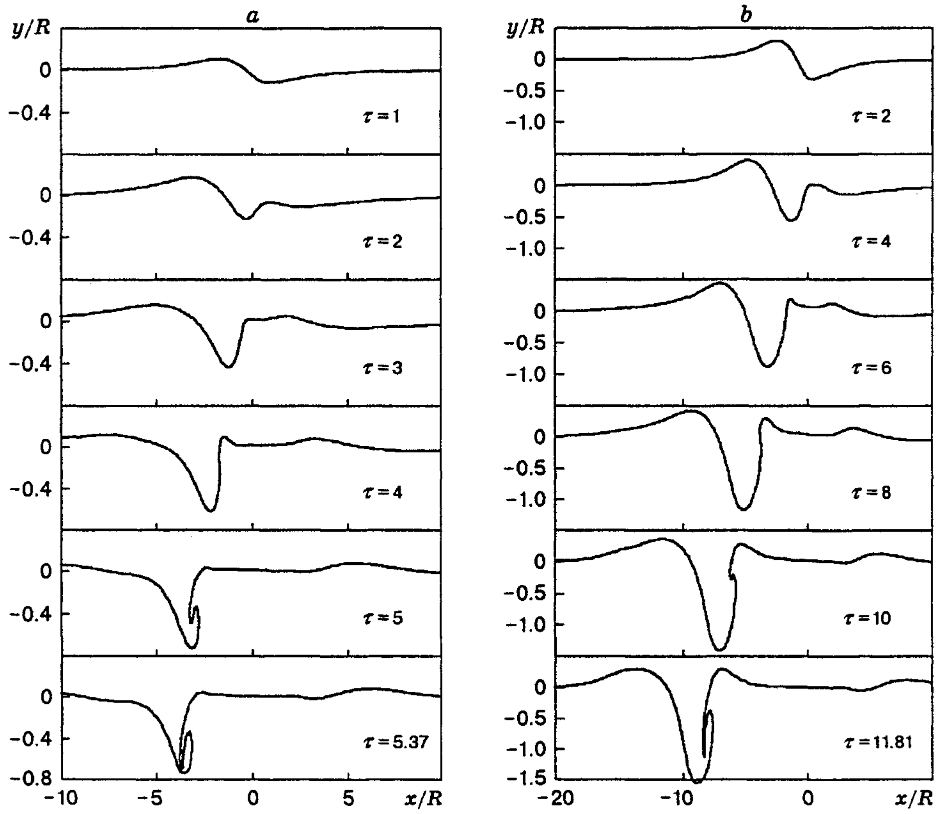

Fig. 1. Free surface shape for Fr = 0.5 (a) and 1 (b).

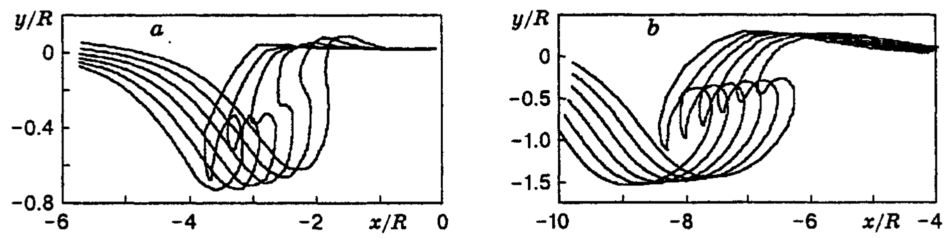

Fig. 2. Wave breaking at successive times: (a) τ = 4.1, 4.35, 4.6, 4.85, 5.1, and 5.37 for Fr = 0.5; (b) τ = 10.55, 10.8, 11.05, 11.3, 11.55, and 11.81 for Fr = 1.

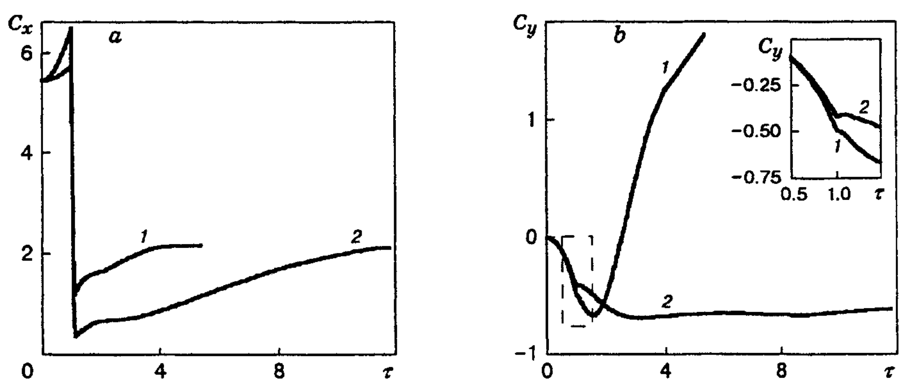

Fig. 3. Coefficients of wave drag (a) and body force (b) of a circular cylinder for Fr = 0.5 (curve 1) and 1 (curve 2).

TABLE 1

T_1	τ_*		
	Fr = 0.5	Fr = 1	Fr = 1.4142
0.05	4.68	10.27	21.92
1.0	5.37	11.81	23.78

the ends of the intervals, from (2.1) we obtain the values of $\overline{V}^n(z)$ at the points $z^n(s_{0j}^*) \in L_0^n$, and then from (2.7) we obtain the overall hydrodynamic characteristics.

4. The method developed previously is employed to solve the problem of the acceleration of a circular cylinder of radius R under the free surface of a heavy liquid ($\rho_* = 1$). The dimensionless parameters of the problem are the Froude number $\mathrm{Fr} = U_0/\sqrt{gR}$, time $\tau = tU_0/R$ (the time until which the cylinder is accelerated $T_1 = TU_0/R$), and the distance from the center of the cylinder to the undisturbed free surface h/R. The following parameter values are selected: $h/R = 2$, Fr = 0.5, 1, 1.4142, $T_1 = 0.05$, and 1.

The computation domain was considered on the segments $[-25, 25]$ for Fr = 0.5, $[-30, 30]$ for Fr = 1 and $[-35, 35]$ at Fr = 1.4142. The number of nodes was 500, 600, and 700, respectively, on the free surface and 60 on the contour. The waves reflected from the boundaries of the computation domain were eliminated by introduction of a damping layer on segments of length $5R$ located at the ends of the calculation interval according to the procedure described in [7]. System (2.4), (2.5) was integrated with respect to time by the Runge–Kutta–Fellberg method of the fifth order of accuracy [8]. The time step $\Delta\tau$ was varied dynamically from 0.05 to 0.01. The value of the derivative $\partial G(s_1, t)/\partial s_1$, the slope of the interface $\theta_1(s_1, t)$ to the x axis, and the integrals entering the expressions for hydrodynamic loads (2.7) were calculated by means of cubic splines. The short-wave instability originating on the free surface was eliminated by the filtration procedure developed in [9]. The system of linear algebraic equations was solved by the overrelaxation method. To increase the calculation accuracy from the time a vertical segment appears on the free surface until complete breaking of the waves τ_*, we employed a redivision using parametric splines.

The process of solution of the problem was checked by means of the integral energy conservation law. Under the above-mentioned assumptions on the number of nodes and magnitude of the step $\Delta\tau$, the energy variation during the computing did not exceed 1%.

Table 1 gives the values of τ_* for which the accelerated motion of the circular cylinder from the quiescent state ($h/R = 2$) leads to wave breaking, and reflects the effect of the quantity T_1 on the time τ_* for various Froude numbers. The natural result is obtained: a decrease in the acceleration time leads to stronger disturbances and earlier time of breaking. An increase in the Froude number Fr leads to an increase in τ_*.

Figures 1 and 2 show the calculated shape of the free surface, and Fig. 3a and b shows the coefficients of wave drag $C_x = 2R_x/\rho_1 RU_0^2$ and lifting force $C_y = 2R_y/\rho_1 RU_0^2$, respectively, for $T_1 = 1$ and Fr = 0.5 and 1. The free surface behaves as follows: initially, an elevation produced by acceleration forms ahead of the contour, and then it begins to decrease and, simultaneously, a surge from the hollow behind the cylinder forms. Further, a vertical segment on the free surface forms, after which wave breaking occurs. The energy of the breaking wave is replenished owing to the decrease in the free surface in the generated hollow. This breaking pattern is observed for all Froude numbers specified above. The wave drag increases with time, and this is typical evidence for nonstationarity of the flow produced by a moving body [10]. The lifting force is negative and nonmonotonic over the entire interval of motion. It should be noted that for Fr = 0.5 there is a segment of positive lifting force, which corresponds to the buoyancy force acting on the cylinder. Another interesting feature in the behavior of the hydrodynamic loads is the absence of monotonicity in a small neighborhood $\tau = 1$, which is explained by discontinuity of the acceleration. There are a number of explicit analytical formulas confirming this fact (see, for example, [11]).

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